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Negaton and positon solutions of the KdV and mKdV hierarchy

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Abstract. We give a systematic classification and a detailed discussion of the structure, motion and scattering of the recently discovered negaton and positon solutions of the Korteweg–de Vries hierarchy. There are two distinct types of negaton solutions which we label $[S^n]$ and $[C^n]$, where $(n+1)$ is the order of the Wronskian used in the derivation. For negatons, the number of singularities and zeros is finite and they show very interesting time dependence. The general motion is in the positive x direction, except for certain negatons which exhibit one oscillation around the origin. In contrast, there is just one type of positon solution, which we label $[\tilde{C}^n]$. For positons, one gets a finite number of singularities for n odd, but an infinite number for even values of n. The general motion of positons is in the negative x direction with periodic oscillations. Negatons and positons retain their identities in a scattering process and their phase shifts are discussed. We obtain a simple explanation of all phase shifts by generalizing the notions of 'mass' and 'centre of mass' to singular solutions. Finally, it is shown that negaton and positon solutions of the KdV hierarchy can be used to obtain corresponding new solutions of the modified KdV hierarchy.

1. Introduction

One of the most studied nonlinear evolution equations in mathematical physics is the Korteweg–de Vries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0\,. \tag{1}$$

It is well known that the KdV equation is completely integrable and gives rise to an infinite number of conservation laws [1–3]. Although a great deal is known about non-singular multi-soliton solutions, singular solutions of the KdV equation have been discussed to a much lesser extent [4–7]. To our knowledge, a comprehensive treatment of singular solutions is not available. All the above solutions as well as more complicated new solutions called negatons and positons can all be obtained from Matveev's recent generalized Wronskian formula for solutions of the KdV equation [5]. This formula makes use of an arbitrary number of solutions of the Schrödinger equation at energies k_i^2 and their derivatives with respect to k_i as inputs. In this paper, we only consider the simplest case of a zero background potential, that is the free particle Schrödinger equation. If only one input solution at energy k^2 is used and it has negative (positive) energy, the resulting KdV solutions are called negatons (positons). Using several input solutions permits the study of scattering. Similar approaches can also be applied to other nonlinear evolution equations [8–10].

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In this paper, we make a systematic classification of negatons and positons for the entire KdV hierarchy and study their structure, motion and interactions. We develop a physical picture underlying negaton and positon solutions which helps to give an intuitive understanding of their x and t dependences. Similarly, we generalize the standard concepts of 'mass' and 'centre of mass' to non-singular solutions, and use them to give simple quantitative explanations for the phase shifts in various scattering processes. We give many figures showing negatons and positons in motion, since this provides a good pictorial grasp of time dependence. In section 2, we present the general formalism and establish notation relevant to solutions of the KdV equation. Section 3 contains a detailed description and classification of negaton solutions. There are two types of negatons $[C^n]$ and $[S^n]$, $n = 0, 1, 2, \ldots$. The singularity patterns and their time dependence are particularly interesting and we describe these in detail. Section 4 contains a description of the structure and motion of positons. There is only one type of positon $[\tilde{C}^n]$, $n = 0, 1, 2, \ldots$ and one has a finite number of singularities for odd values of n. The motion for this case is physically quite different from the negaton case. This can be understood, since one is using trigonometric functions instead of hyperbolic functions. Scattering of negatons and positons is treated in section 5. It is found that negatons and positons emerge from an interaction preserving their identity, but often with a shift in phase. In section 6, we discuss the positon and negaton solutions of the modified Korteweg–de Vries (mKdV) equation. Finally in section 7 we discuss the positon and negaton solutions of the entire KdV and mKdV hierarchy. Some open problems and concluding remarks are given in section 8.

2. General formalism and notation

Solutions of equation (1) can be systematically obtained from solutions of the free particle Schrödinger equation ($\hbar = 2m = 1$):

$$-\frac{\mathrm{d}^2\phi_i}{\mathrm{d}x^2} = E_i\phi_i\,. \tag{2}$$

For $E_i = -k_i^2 < 0$, a convenient choice of independent solutions $\phi_i(k_i x)$ is $\sinh k_i x$ and $\cosh k_i x$, whereas for $E_i = \tilde{k}_i^2 > 0$, the independent solutions $\phi_i(\tilde{k}_i x)$ can be chosen to be the trigonometric functions $\sin \tilde{k}_i x$ and $\cos \tilde{k}_i x$. (It can be shown that nothing new is obtained by taking more general linear combinations.) We consider solutions of (1) of the form [5, 11]

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\ln W = 2\frac{(W'^2 - WW'')}{W^2} \tag{3}$$

where $W = W(\phi_1, \ldots, \phi_n)$ is the Wronskian determinant composed of $\phi_i(\theta_i)$. Here θ_i stands for

$$\theta_i = k_i(x + \xi_i(k_i) - 4k_i^2 t) \tag{4}$$

for $E_i < 0$, and

$$\tilde{\theta}_i = \tilde{k}_i(x + \tilde{\xi}_i(\tilde{k}_i) + 4\tilde{k}_i^2 t) \tag{5}$$

for $E_i > 0$. $\xi_i(\tilde{\xi}_i)$ are arbitrary functions of $k_i(\tilde{k}_i)$, $i = 1, 2, \ldots, n$.

For clarity, let us first focus on solutions of the KdV equation which come from negative energy solutions of equation (2). The simplest choice is to have a Wronskian of order 1. Here, we have two types of solutions which can be either $\cosh\theta$ or $\sinh\theta$. For $\phi = \cosh\theta$, one gets $u(x,t) = -2k^2\mathrm{sech}^2\theta$, which we denote by $[C^0]$. This is the usual non-singular one-soliton solution moving to the right (along the positive x direction) with speed $4k^2$. For $\phi = \sinh\theta$, one gets $u(x,t) = 2k^2 \mathrm{cosech}^2\theta$, which we denote by $[S^0]$. This is the simplest singular solution of the KdV equation [1]. There is just one singularity at $x = 4k^2 t - \xi(k)$ moving to the right at speed $4k^2$. $[C^0]$ and $[S^0]$ are called negaton solutions of order 0.

For Wronskians of order 2 there are three types of solutions:

(i) $\phi_1 = \cosh\theta_1 \qquad \phi_2 = \cosh\theta_2$

(ii) $\phi_1 = \cosh\theta_1 \qquad \phi_2 = \sinh\theta_2$

(iii) $\phi_1 = \sinh\theta_1 \qquad \phi_2 = \sinh\theta_2$

where θ_1, θ_2 are given by (4), and correspond to speeds $4k_1^2$ and $4k_2^2$, respectively. It is easy to check [5] that case (ii) is the well known finite two-soliton solution of the KdV equation, whereas cases (i) and (iii) correspond to solutions with one singularity.

Of particular interest to us is the situation where $k_1 = k$ and $k_2 = k + \epsilon$ with ϵ tending to zero. In order to get a non-trivial solution, it is necessary to make the choice $\xi_1(k) = \xi_2(k) = \xi(k)$ in (4). For cases (i) and (iii), W, W' and W'' are all $\mathrm{O}(\epsilon)$. Thus, from (3), $u(x,t)$ is $\mathrm{O}(\epsilon^0)$. This is a new solution [5] of the KdV equation which does not vanish as $\epsilon \to 0$. For case (ii), however, $W \to$ constant as $\epsilon \to 0$ and no new solutions result.

The new solutions coming from cases (i) and (iii) will be denoted by $[C]$ and $[S]$, respectively and are called negaton solutions of order 1. More explicitly, for these cases $\phi_2(k+\epsilon) = \phi_1(k) + \epsilon\partial_k\phi_1(k) + \mathrm{O}(\epsilon^2)$, and the Wronskian is

$$W(\phi_1,\phi_2) \equiv W(\phi_1, \phi_1 + \epsilon\partial_k\phi_1) = \epsilon W(\phi_1, \partial_k\phi_1)\,. \tag{6}$$

The multiplicative constant ϵ does not play any role in obtaining $u(x,t)$ using (3) and can be dropped from the Wronskian. For the $[C]$ case,

$$W \equiv W(\cosh\theta, \partial_k\cosh\theta) = k\gamma + \cosh\theta\sinh\theta \tag{7}$$

where

$$\gamma \equiv \partial_k\theta = x + \xi(k) + k\partial_k\xi(k) - 12k^2 t\,. \tag{8}$$

Similarly, for the $[S]$ case, the Wronskian reads:

$$W \equiv W(\sinh\theta, \partial_k\sinh\theta) = -k\gamma + \cosh\theta\sinh\theta\,. \tag{9}$$

Although we have so far only discussed negaton Wronskians of order 1 and 2, the above results can be readily extended to Wronskians of any higher order. A straightforward extension of (6) yields a Wronskian determinant of order $(n+1)$:

$$W = W(\phi, \partial_k\phi, \ldots, \partial_k^n\phi)\,. \tag{10}$$

This is a special case of the generalized Wronskian formula given by Matveev [5]. If the Wronskian of (10) with $\phi = \cosh\theta$ is used in (3), the resulting KdV solution is called a negaton $[C^n]$ of order n with $n = 0, 1, 2, \ldots$. Similarly, the choice $\phi = \sinh\theta$ yields a negaton $[S^n]$ of order n. To summarize, the negaton corresponding to equation (10) has the physical interpretation of merging $(n+1)$ solutions ϕ of the free particle Schrödinger equation all with wave numbers near k and identical phases $\xi(k)$.

The entire discussion given above for negatons also holds for positive energy solutions of the Schrödinger equation. The solutions of the KdV equation resulting from the choices $\phi = \cos\tilde\theta$ and $\phi = \sin\tilde\theta$ are called positons of order n [5] and are denoted by $[\tilde C^n]$ and $[\tilde S^n]$, respectively.

An important difference between positons and negatons is that the positons $[\tilde C^n]$ and $[\tilde S^n]$ are not independent. In fact, the choice $\tilde k\xi_i = \pi/2$ in $\tilde\theta_i$ in (5) transforms $[\tilde C^n]$ into

$[\tilde{S}^n]$. On the other hand, negatons $[C^n]$ and $[S^n]$ are physically different. As we shall see, they usually have a different number of singularities for the same value of n. However, one can mathematically transform $[C^n]$ into $[S^n]$ by the unphysical imaginary choice of phase $k\xi_i = \mathrm{i}\pi/2$.

It is also interesting to observe that positon solutions can be obtained from the corresponding negaton solutions via the change $k \to \mathrm{i}\tilde{k}$. Note that the x and t dependence of all solutions comes from θ $(\tilde{\theta})$ and derivatives of θ $(\tilde{\theta})$ with respect to k $(\tilde{k})$. From now on, our discussion is based on making the simplest choice $\xi(k) = 0$ in (4), (5) and (8). It is important to observe that under the transformations $x \to -x$ and $t \to -t$, θ $(\tilde{\theta})$ and all derivatives with respect to k $(\tilde{k})$ change sign. As a result, the Wronskian W in (10) has the property $W(-x, -t) = \pm W(x, t)$. Thus, for all negaton or positon solutions, it follows that $u(x, t) = u(-x, -t)$, and it is sufficient just to consider the behaviour at either negative or positive values of t. In particular, at time $t = 0$, all solutions are symmetric $u(x, 0) = u(-x, 0)$.

The 'mass' and 'centre of mass' of any solution $u(x, t)$ of the KdV equation are useful concepts in analysing the behaviour of negatons and positons. Here, $u(x, t)$ is identified with a linear mass distribution, and the total mass is given by

$$M \equiv \int_{-\infty}^{\infty} u(x, t)\,\mathrm{d}x\,. \tag{11}$$

This definition is only useful for non-singular solutions $u(x, t)$. However, it is easy to obtain an alternative, more generally applicable definition. Using equation (3) for non-singular $u(x, t)$, the total mass can be written as

$$M = -2[W'/W]_{-\infty}^{+\infty}\,. \tag{12}$$

We will use (12) as the definition of the mass M, an expression which is well defined for both non-singular as well as singular solutions $u(x, t)$. M is a constant of the motion. Note that our definition is equivalent to the $x + \mathrm{i}\epsilon$ regularization procedure suggested in [7]. Also, the position of the centre of mass is given by

$$x_{\mathrm{CM}} \equiv \frac{1}{M}\int_{-\infty}^{\infty} x u(x, t)\,\mathrm{d}x\,. \tag{13}$$

Again, using equation (3), it is possible to rewrite the expression for the centre-of-mass position,

$$x_{\mathrm{CM}} = \frac{1}{M}\left[-2x\frac{W'}{W} + \ln W^2\right]_{-\infty}^{+\infty}. \tag{14}$$

This definition can be used for all solutions $u(x, t)$. The centre of mass moves at a constant speed [1].

3. Structure and motion of negatons

In this section, we describe the x and t dependences of negatons $u(x, t)$ corresponding to Wronskians of different orders. A summary of some characteristics and properties of the simplest negatons is given in table 1.

Wronskians of order 1. Here, one has the familiar results:

$$[C^0] \qquad u(x, t) = -2k^2\mathrm{sech}^2\theta \tag{15}$$

$$[S^0] \qquad u(x, t) = 2k^2\mathrm{cosech}^2\theta\,. \tag{16}$$

Table 1. Various characteristics of $[C^n]$ and $[S^n]$ negatons for $n = 0, 1, 2, 3$.

Order of Wronskian	Negaton type	Wronskian $W(x,t)$	Dominant term in Wronskian at $x \to \pm\infty$	Poles of $u(x,t)$	Zeros of $u(x,t)$
1	$[C^0]$	$\cosh\theta$	$\cosh\theta$	0	0
1	$[S^0]$	$\sinh\theta$	$\sinh\theta$	1	0
2	$[C]$	$k\gamma + \cosh\theta\sinh\theta$	$\cosh\theta\sinh\theta$	1	2
2	$[S]$	$-k\gamma + \sinh\theta\cosh\theta$	$\sinh\theta\cosh\theta$	1	2
3	$[C^2]$	equation (19)	$\cosh^2\theta\sinh\theta$	1	4
3	$[S^2]$	equation (20)	$\sinh^2\theta\cosh\theta$	2	4
4	$[C^3]$	equation (21)	$\cosh^2\theta\sinh^2\theta$	2	6
4	$[S^3]$	equation (22)	$\sinh^2\theta\cosh^2\theta$	2	6

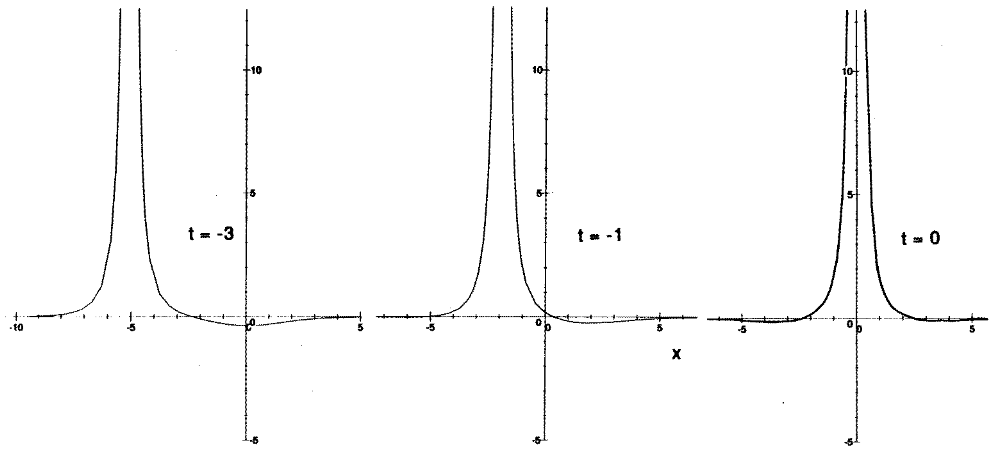

Figure 1. The shape and motion of the $[C]$ negaton as given by (17) for $k = 0.5$ and $\xi(k) = 0$.

Both negatons move with constant speed $4k^2$, and their shape remains unchanged. The 'masses' of both the $[C^0]$ and $[S^0]$ negatons as given by (12) are $-4k$.

Wronskians of order 2. The explicit Wronskians are given in table 1 and the corresponding KdV solutions exhibit very interesting behaviour. The $[C]$ negaton is given by

$$[C] \qquad u(x,t) = \frac{8k^2\cosh\theta(\cosh\theta - k\gamma\sinh\theta)}{(\cosh\theta\sinh\theta + k\gamma)^2}\,. \tag{17}$$

Its shape and motion is shown in figure 1. It has one singularity corresponding to the zero of its Wronskian (see table 1). At any fixed time t, the dominant term in the Wronskian at $x \to \pm\infty$ is $\cosh\theta\sinh\theta$. Therefore, one expects the Wronskian necessarily to have an odd number of zeros. For this case, there is just one zero giving rise to the singular behaviour $u(x,t) \propto \frac{2}{(x-x_P(t))^2}$. At large negative time t, since the main term in the Wronskian is $\cosh\theta\sinh\theta$, one gets a $[C]$ negaton composed of a 'soliton' $[C^0]$ (corresponding to the $\cosh\theta$ factor) with a singularity $[S^0]$ on the left (corresponding to the $\sinh\theta$ factor). This structure immediately suggests that the mass of the $[C]$ negaton should be $-8k$, and a computation using (12) confirms this to be the case. The 'centre of mass' of the negaton is approximately halfway between the singularity and the minimum of the 'soliton', and it moves

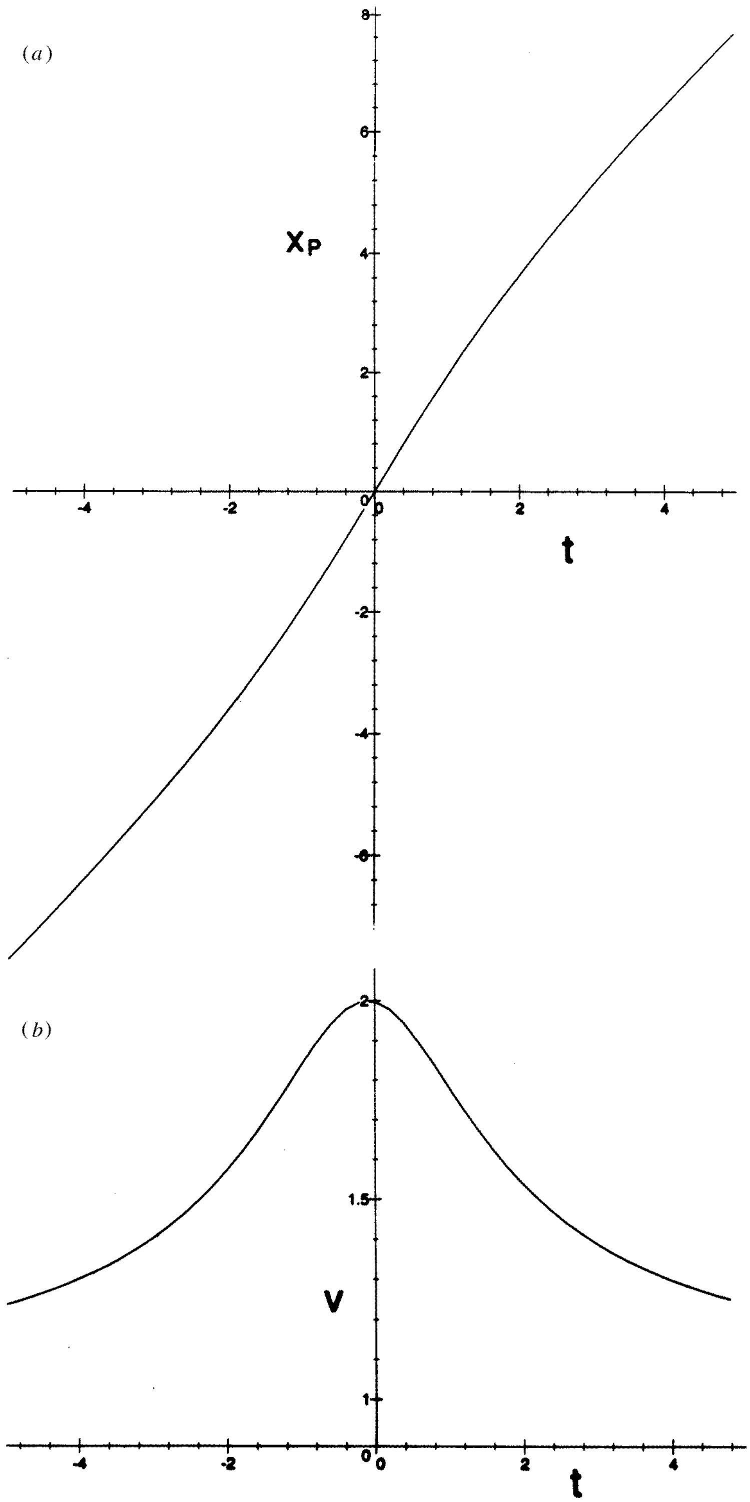

Figure 2. The (*a*) position x_P and (*b*) velocity v of the singularity of the [*C*] negaton of figure 1 as a function of time.

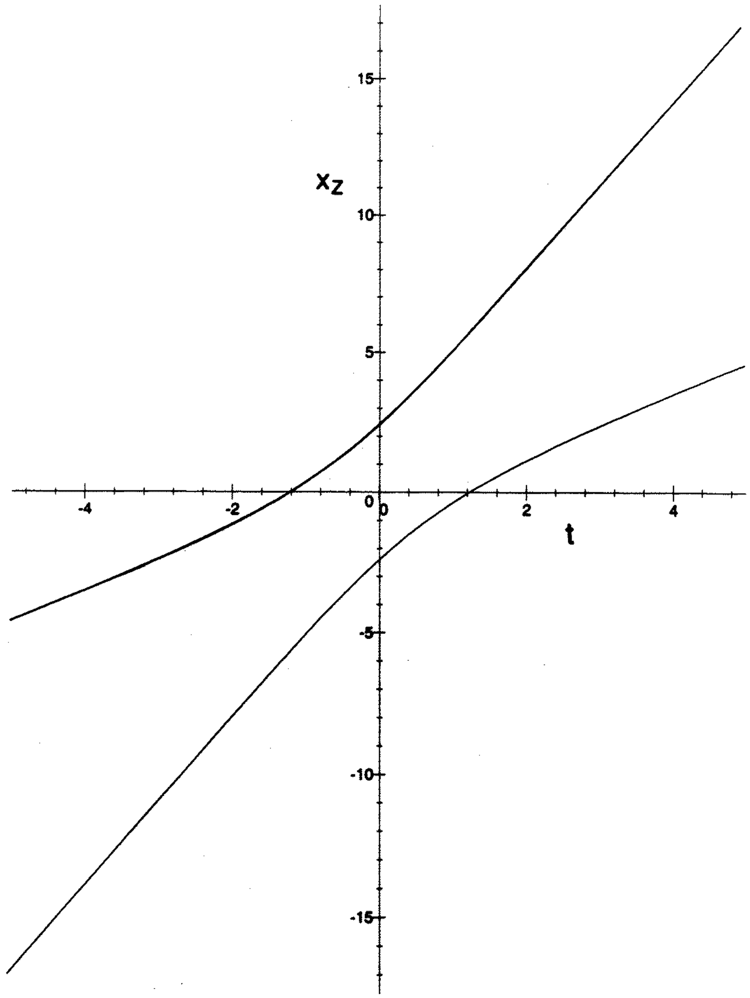

Figure 3. The positions x_Z of the two zeros of the $[C]$ negaton of figure 1 as a function of time.

with a constant speed $4k^2$. The motion of the pole $x_P(t)$ is shown in figure 2, which shows its position and speed. Note that the pole has an asymptotic speed $4k^2$, which is expected since the Wronskian just becomes a function of θ at $t \to \pm\infty$. $u(x, t)$ also has two zeros coming from the numerator of equation (17). These two zeros move as shown in figure 3.

Similarly, the $[S]$ negaton is

$$[S] \qquad u(x,t) = \frac{-8k^2 \sinh\theta(\sinh\theta - k\gamma\cosh\theta)}{(\sinh\theta\cosh\theta - k\gamma)^2}\,. \tag{18}$$

This is similar in form to the $[C]$ negaton with $\sinh\theta$ and $\cosh\theta$ exchanged. As can be seen in figure 4, at large negative time, the $[S]$ negaton is a singularity $[S^0]$ (corresponding to the $\sinh\theta$ factor in W) along with a 'soliton' $[C^0]$ on the left (corresponding to the $\cosh\theta$ factor in W). It has a mass $-8k$. The $[S]$ negaton shows an interesting oscillation of its singularity near $x = t = 0$. The singularity moves continuously to the right and comes to a momentary halt at a positive value of x. It then reverses its direction of motion, goes

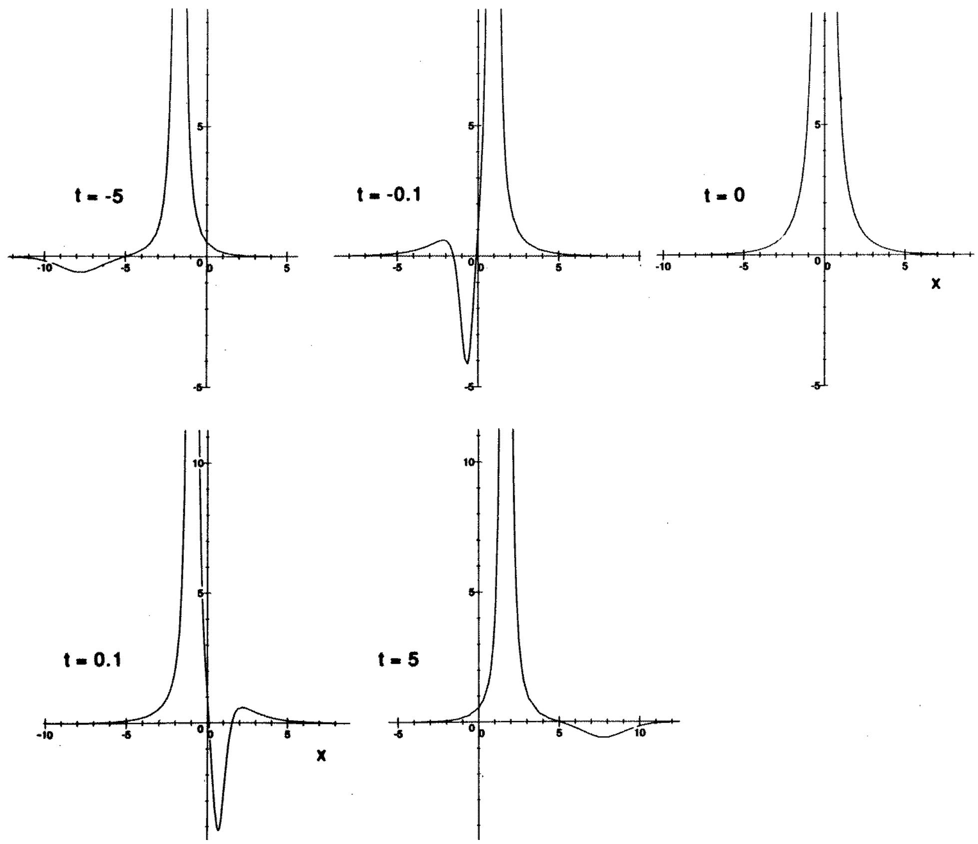

Figure 4. The shape and motion of the $[S]$ negaton as given by (18) for $k = 0.5$ and $\xi(k) = 0$.

past the origin at time $t = 0$ with infinite instantaneous velocity and again comes to a halt at a negative value of x. Thereafter, the motion of the singularity is continuously in the positive x direction, with an asymptotic speed $4k^2$. The motion of the $[S]$ negaton and the time dependence of its singularity and its two zeros are shown in figures 4, 5 and 6.

Wronskians of order 3–5. The Wronskians of order 3 and 4 are given by:

$$[C^2] \qquad W = \tfrac{1}{2}\sinh 3\theta + \sinh\theta\left(\tfrac{1}{2} + 4k^2\gamma^2\right) - 2k\gamma\cosh\theta + 48k^3t\cosh\theta \tag{19}$$

$$[S^2] \qquad W = \tfrac{1}{2}\cosh 3\theta - \cosh\theta\left(\tfrac{1}{2} + 4k^2\gamma^2\right) + 2k\gamma\sinh\theta - 48k^3t\sinh\theta \tag{20}$$

$$\begin{aligned}[C^3] \qquad W = {} & \tfrac{3}{2}\cosh 4\theta + \cosh 2\theta(-24k^2\gamma^2 + 576k^4\gamma t) \\ & + \sinh 2\theta(16k^3\gamma^3 + 12k\gamma - 384k^3t) - 16k^4\gamma^4 - 12k^2\gamma^2 \\ & - 192k^4\gamma t - 6912k^6t^2 - \tfrac{3}{2}\end{aligned} \tag{21}$$

$$\begin{aligned}[S^3] \qquad W = {} & \tfrac{3}{2}\cosh 4\theta + \cosh 2\theta(24k^2\gamma^2 - 576k^4\gamma t) \\ & - \sinh 2\theta(16k^3\gamma^3 + 12k\gamma - 384k^3t) - 16k^4\gamma^4 - 12k^2\gamma^2 \\ & - 192k^4\gamma t - 6912k^6t^2 - \tfrac{3}{2}\,.\end{aligned} \tag{22}$$

For any Wronskian, the corresponding KdV solution is readily obtained from (3). The number of zeros and singularities is shown in table 1. In particular, the $[S^4]$ negaton has

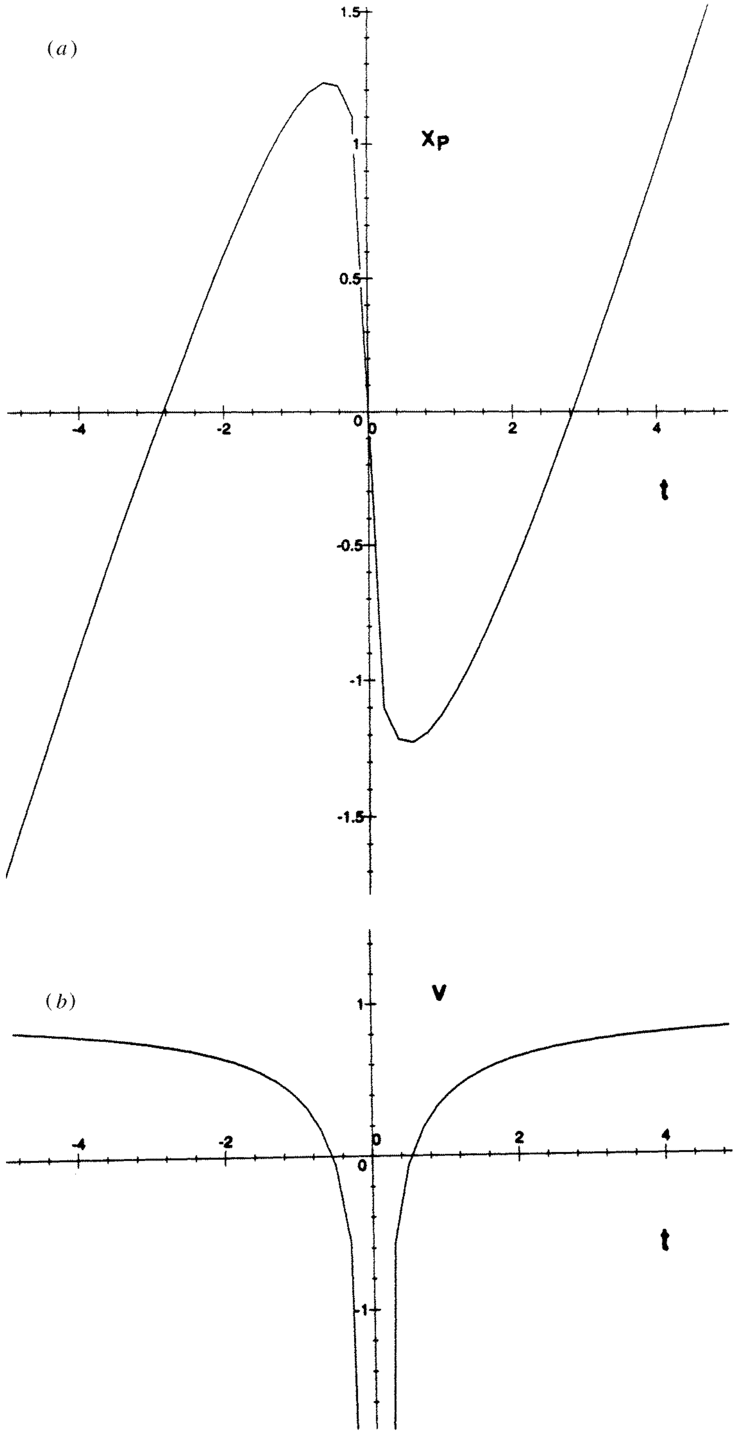

Figure 5. The (*a*) position x_P and (*b*) velocity v of the singularity of the [*S*] negaton of figure 4 as a function of time.

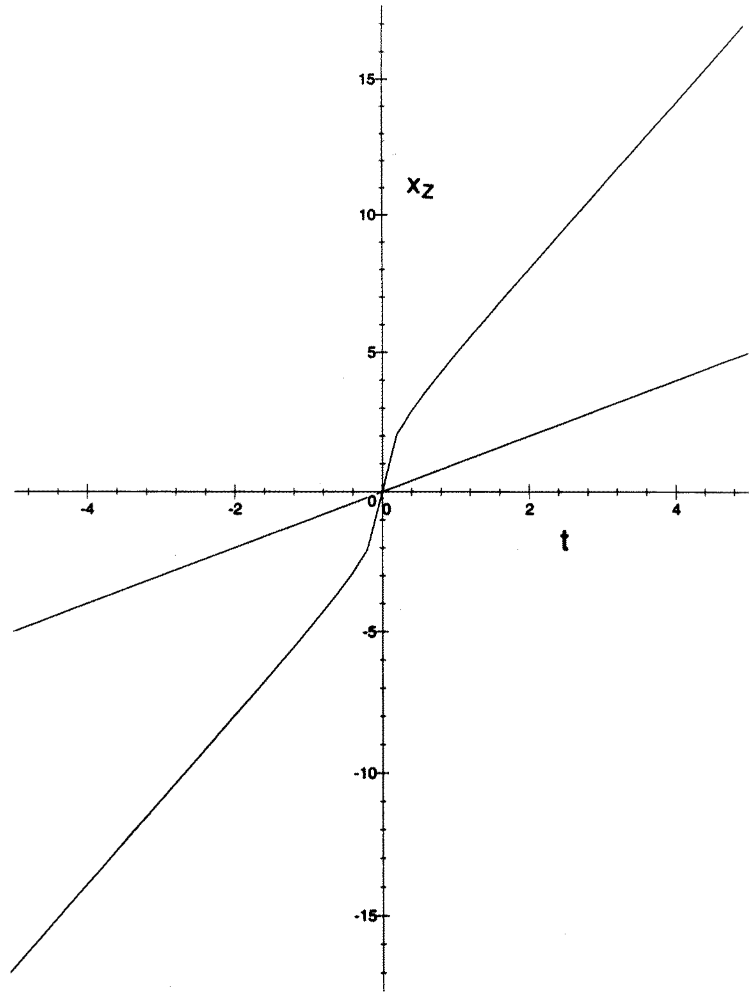

Figure 6. The positions of the two zeros x_Z of the $[S]$ negaton of figure 4 as a function of time.

eight zeros and three singularities. Their motion is shown in figures 7 and 8. Again, note that at large negative time, the $[S^4]$ negaton has a dominant term in the Wronskian '$\sinh\theta \cosh\theta \sinh\theta \cosh\theta \sinh\theta$', which gives the structure 'singularity-soliton-singularity-soliton-singularity' seen in figure 7. The two leading singularities show an oscillation around $x = t = 0$, but the third one does not.

Wronskians of order $(n+1)$. At this stage, let us generalize the discussion to Wronskian determinants of arbitrary order $(n+1)$, $n = 0, 1, 2, \ldots$. For any given negaton, the number of singularities and the number of zeros are both finite. These numbers become steadily larger as the order of the Wronskian increases. The number of singularities and the number of zeros remains constant in time and hence characterize a given negaton. The dominant terms in the Wronskians at $x \to \pm\infty$ are given in table 1. They follow a simple rule, which tells whether there are an odd or an even number of negaton singularities. Based on these

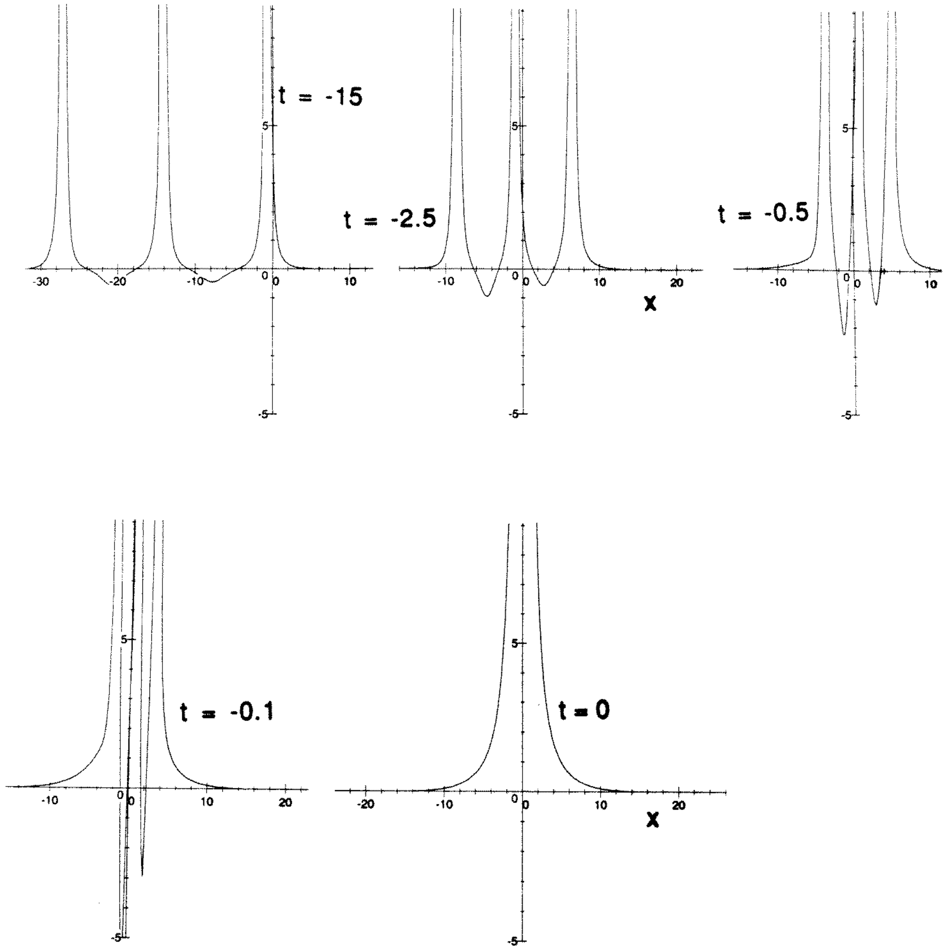

Figure 7. The shape and motion of the $[S^4]$ negaton for $k = 0.5$ and $\xi(k) = 0$. Note that although only four zeros are manifestly visible for the scales used in the figure, there are indeed a total of eight zeros corresponding to the formula $2n$ discussed in section 3.

considerations, we expect the following general formulae for the number of singularities:

$$\begin{array}{lll} [C^n] & (n+1)/2 \ \text{ for } n \text{ odd} & n/2 \ \text{ for } n \text{ even} \\ [S^n] & (n+1)/2 \ \text{ for } n \text{ odd} & (n+2)/2 \ \text{ for } n \text{ even}. \end{array} \tag{23}$$

Similar considerations show that there are $2n$ zeros for both $[C^n]$ and $[S^n]$ negatons. At least for the choice $\xi(k) = 0$, we have checked that the number of zeros remains unchanged in time. It is easy to show that the mass of the $[C^n]$ and $[S^n]$ negatons is $-4(n+1)k$, and the centre of mass moves with constant speed $4k^2$.

It is interesting to analyse some features of negatons at time $t = 0$. The Wronskian for $[S^n]$ has the flat behaviour $(kx)^{(n+1)(n+2)/2}$ near $x = 0$ and consequently the KdV solution has the behaviour $u(x, 0) \propto \frac{(n+1)(n+2)}{x^2}$. Likewise, $[C^n]$ has the singular behaviour $u(x, 0) \propto \frac{n(n+1)}{x^2}$ at small x. Note that the time $t = 0$ is very special, since all negaton singularities merge at $x = 0$. At any other time t, the singularities are separated, each exhibiting a $\frac{2}{(x - x_P(t))^2}$ behaviour. For any given negaton, the separation between singularities

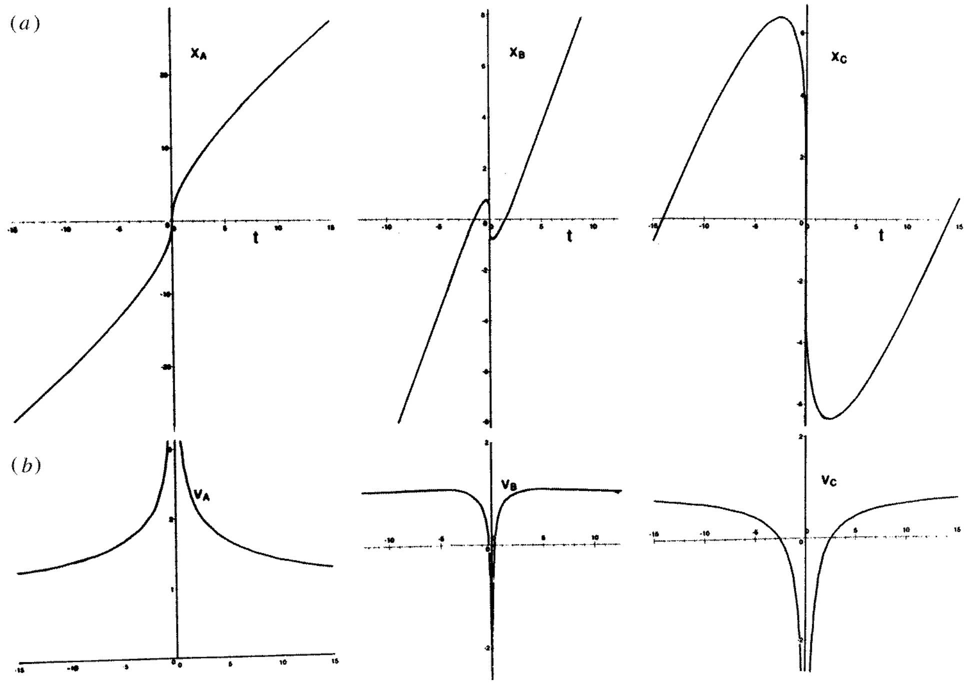

Figure 8. The (*a*) position x_P and (*b*) velocity v of the three singularities of the $[S^4]$ negaton of figure 7 as a function of time.

becomes constant at large t, since all singularities asymptotically move at the same speed $4k^2$.

4. Structure and motion of positons

The analysis for positons is somewhat different than for negatons since there is only one type labelled by $[\tilde{C}^n]$ and trigonometric functions are involved. A summary of properties is given in table 2. The simplest positon is

$$[\tilde{C}^0] \qquad W = \cos\tilde{\theta} \qquad u(x,t) = 2\tilde{k}^2 \sec^2\tilde{\theta} \qquad \tilde{\theta} = \tilde{k}(x + 4\tilde{k}^2 t) \tag{24}$$

which moves to the left at a constant speed $4\tilde{k}^2$. It has an infinite number of singularities, and in fact this property holds for positons of any even order n [7]. In contrast, for odd values of n, the number of singularities is finite but the number of zeros is infinite. The positon of order 1 $[\tilde{C}]$ has been extensively discussed [5, 7].

$$[\tilde{C}] \qquad u(x,t) = \frac{8\tilde{k}^2 \cos\tilde{\theta}(\cos\tilde{\theta} + \tilde{k}\tilde{\gamma}\sin\tilde{\theta})}{(\sin\tilde{\theta}\cos\tilde{\theta} + \tilde{k}\tilde{\gamma})^2} \tag{25}$$

where $\tilde{\gamma} \equiv \partial_{\tilde{k}}\tilde{\theta}$. In order to compare the motion of positons with negatons, we show the motion of the singularity of the $[\tilde{C}]$ positon in figure 9. The graph shows several more or less straight sections with periodic jumps. The straight sections have an average slope $8\tilde{k}^2$ corresponding to the difference of the two characteristic speeds $4\tilde{k}^2$ and $12\tilde{k}^2$ contained in the quantities $\tilde{\theta}$ and $\tilde{\gamma}$ respectively in the Wronskian. The jumps occur at

Table 2. Various characteristics of $[\tilde{C}^n]$ positons for $n = 0, 1, 2, 3$.

Order of Wronskian	Positon type	Wronskian $W(x, t)$	Poles of $u(x, t)$
1	$[\tilde{C}^0]$	$\cos\tilde{\theta}$	∞
2	$[\tilde{C}]$	$-\tilde{k}\tilde{\gamma} - \cos\tilde{\theta}\sin\tilde{\theta}$	1
3	$[\tilde{C}^2]$	equation (26)	∞
4	$[\tilde{C}^3]$	equation (27)	2

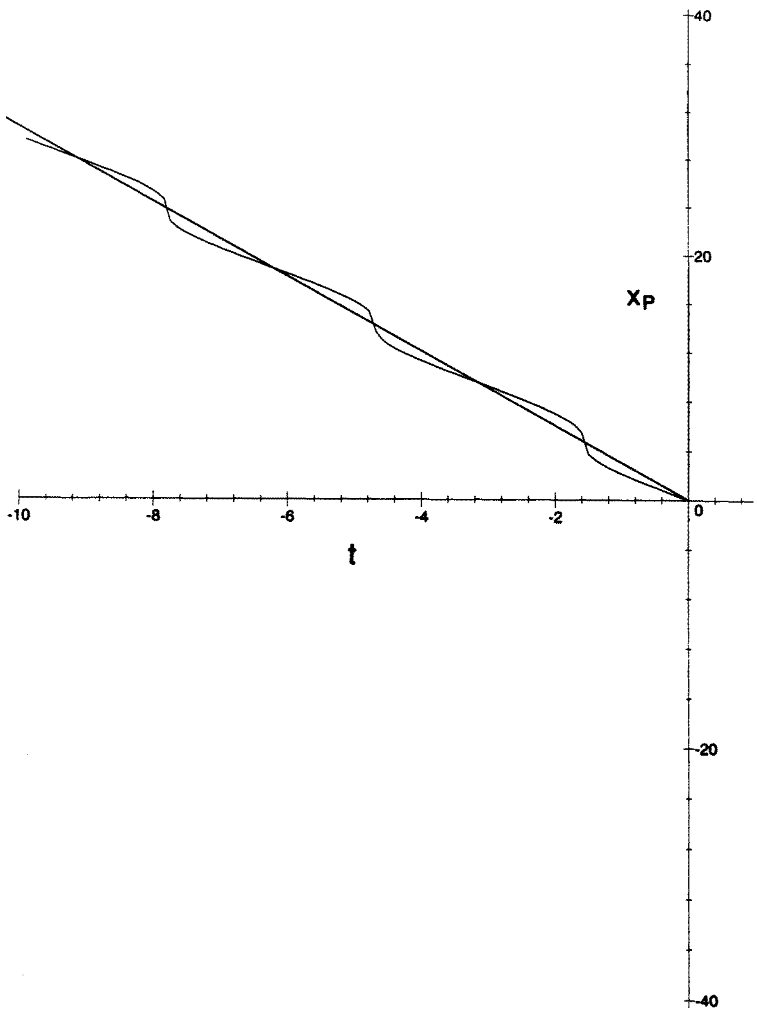

Figure 9. The position x_P of the singularity of the $[\tilde{C}]$ positon with $\tilde{k} = 0.5$ as a function of time. Also shown is the line $x_P = -12\tilde{k}^2 t$, with a slope $-12\tilde{k}^2$ which corresponds to the average speed of the singularity. The straight sections have a slope $-8\tilde{k}^2$.

times $(2m + 1)\pi/(16\tilde{k}^3)$ for $m = 0, \pm 1, \ldots,$ and give rise to infinite speeds. Alternatively, the motion of the singularity can also be described as oscillations around an average constant speed $12\tilde{k}^2$. A more detailed description of this motion and an extension to other even values of n can be found in [7]. For completeness, we give expressions for the Wronskians for $[\tilde{C}^2]$ and $[\tilde{C}^3]$ positons:

$$[\tilde{C}^2] \qquad W = \tfrac{1}{2}\sin 3\tilde{\theta} + \sin\tilde{\theta}\left(\tfrac{1}{2} - 4\tilde{k}^2\tilde{\gamma}^2\right) - 2\tilde{k}\tilde{\gamma}\cos\tilde{\theta} - 48\tilde{k}^3 t\cos\tilde{\theta} \tag{26}$$

$$[\tilde{C}^3] \qquad W = \cos 2\tilde{\theta}(-24\tilde{k}^2\tilde{\gamma}^2 - 576\tilde{k}^4\tilde{\gamma}t) + \sin 2\tilde{\theta}(-16\tilde{k}^3\tilde{\gamma}^3 + 12\tilde{k}\tilde{\gamma} + 384\tilde{k}^3 t) - \tfrac{3}{2}\cos 4\tilde{\theta} + 16\tilde{k}^4\tilde{\gamma}^4 - 12\tilde{k}^2\tilde{\gamma}^2 + 192\tilde{k}^4\tilde{\gamma}t - 6912\tilde{k}^6 t^2 + \tfrac{3}{2}\,. \tag{27}$$

The mass of any positon with odd n is zero. This follows from (12) since all Wronskians of odd order n have a power-like behaviour for $x \to \pm\infty$.

5. Scattering of negatons and positons

Now that we have classified the various types of negaton and positon solutions of the KdV equation and studied their individual structures and motions, we proceed to a discussion of scattering. For simplicity, we restrict our attention to processes involving two incident objects (negatons or positons) with wave numbers k_1 and $k_2 > k_1$. As might be expected from previous work, all these objects emerge from the scattering process preserving their identity but often acquiring a phase shift [1, 5].

Table 3. Various possibilities for the scattering of two negatons of order 0 and wave numbers k_1 and $k_2 > k_1$. The quantities ϕ_i and χ_i are defined in equation (28).

State at $t \to \infty$	Number of poles	Wronskian $W(x, t)$
$[C^0][C^0]$	0	$W(\phi_1, \chi_2)$
$[S^0][S^0]$	2	$W(\chi_1, \phi_2)$
$[C^0][S^0]$	1	$W(\phi_1, \phi_2)$
$[S^0][C^0]$	0	$W(\chi_1, \chi_2)$

Negaton–negaton scattering. The simplest situation is the scattering of two negatons of order 0. There are four possibilities which are shown in table 3. Contained therein is the standard non-singular soliton–soliton case $[C^0][C^0]$ resulting from the Wronskian $W(\phi_1, \chi_2)$, where

$$\phi_i \equiv \cosh\theta_i \qquad \chi_i \equiv \sinh\theta_i \qquad \theta_i = k_i x - 4k_i^3 t \qquad i = 1, 2, \ldots. \tag{28}$$

In general, for N solitons, the asymptotic solution at $t \to \pm\infty$ is

$$u(x, t) = \sum_{i=1}^{N} -2k_i^2 \operatorname{sech}^2(k_i x - 4k_i^3 t \pm \Delta_i)$$

and the phase shifts are well known [1, 2, 12] to be

$$\mathrm{e}^{2\Delta_n} = \prod_{m=1(m\neq n)}^{N} \left|\frac{k_n - k_m}{k_n + k_m}\right|^{\operatorname{sgn}(n-m)}. \tag{29}$$

For our case of $N = 2$, one gets $\Delta_1 = \delta$, $\Delta_2 = -\delta$, where

$$\delta \equiv \frac{1}{2}\ln\left[\frac{k_2 + k_1}{k_2 - k_1}\right]. \tag{30}$$

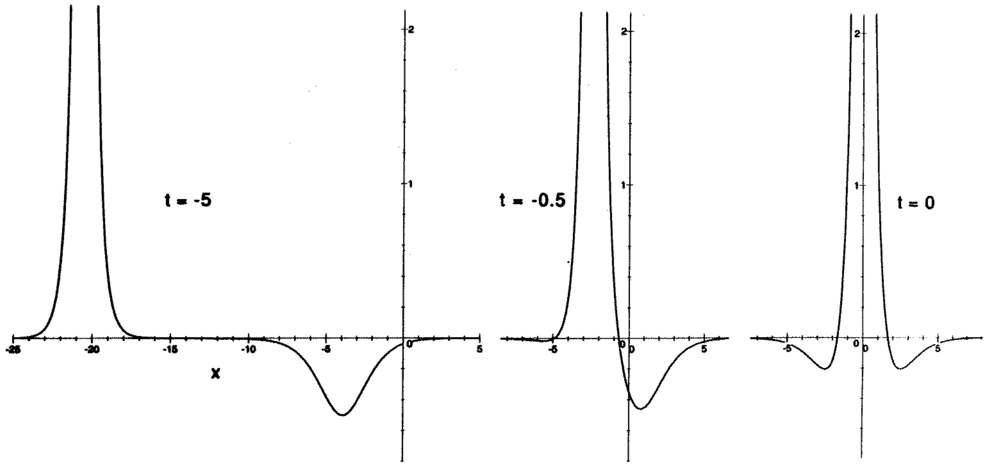

Figure 10. Scattering of a soliton $[C^0]$ with wave number $k_1 = 0.5$ and a $[S^0]$ negaton with wave number $k_2 = 1.0$.

Table 4. Scattering of a negaton of order 0 with a negaton of order 1.

State at $t \to \infty$	Number of poles	Wronskian $W(x, t)$
$[C][S^0]$	2	$W(\phi_1, \partial_{k_1}\phi_1, \chi_2)$
$[C^0][C]$	1	$W(\phi_1, \chi_2, \partial_{k_2}\chi_2)$
$[S][C^0]$	1	$W(\chi_1, \partial_{k_1}\chi_1, \phi_2)$
$[S^0][S]$	2	$W(\chi_1, \phi_2, \partial_{k_2}\phi_2)$
$[C][C^0]$	1	$W(\phi_1, \partial_{k_1}\phi_1, \phi_2)$
$[C^0][S]$	1	$W(\phi_1, \phi_2, \partial_{k_2}\phi_2)$
$[S][S^0]$	2	$W(\chi_1, \partial_{k_1}\chi_1, \chi_2)$
$[S^0][C]$	2	$W(\chi_1, \chi_2, \partial_{k_2}\chi_2)$

The general condition resulting from uniform motion of the overall centre of mass of a system is

$$\sum_{i=1}^{N} \frac{M_i \Delta_i}{k_i} = 0 . \tag{31}$$

Our results for Δ_1 and Δ_2 are consistent with (31) since $M_1 = -4k_1$ and $M_2 = -4k_2$. Note that since $k_2 > k_1$, the Wronskian $W(\chi_1, \phi_2)$ produces a solution $[S^0][S^0]$ with two singularities at large values of time. The case of $[C^0][S^0]$ scattering is shown in figure 10. We have checked from the graphs that the phase shifts are the same as in (30). Indeed all four entries in table 3 are found to have the same phase shifts. This result is very plausible since, as mentioned in section 2, C-type and S-type negatons of any given order are related to each other via an unphysical choice of phase $k\xi = \mathrm{i}\pi/2$, but this does not affect the scattering phase shift calculation.

Next consider the scattering of a negaton of order 0 with a negaton of order 1. Here one has the eight possibilities shown in table 4. As expected, all situations have the same phase shifts. More specifically, if one considers the scattering of a soliton $[C^0]$ with wave number k_1 and a negaton $[S]$ with wave number $k_2 > k_1$, the Wronskian is $W(\phi_1, \phi_2, \partial_{k_2}\phi_2)$ and the soliton gets phase shifted by $\Delta_1 = 2\delta$, where δ is given in (30). This corresponds to the special case of $N = 3$ and $k_3 \to k_2$ in the general formula (29), as might be expected from our physical picture of a negaton. In contrast, the negaton $[S]$ gets phase shifted by $\Delta_2 = -\delta$. This follows from the centre-of-mass condition (31) with masses $M_1 = -4k_1$ and $M_2 = -8k_2$ which were computed before.

It is clear that we can extend the above discussion to the scattering of two negatons of order 1. For two $[C]$ negatons, the Wronskian is $W(\phi_1, \partial_{k_1}\phi_1, \phi_2, \partial_{k_2}\phi_2)$ which can be expanded to give:

$$
\begin{aligned}
W_{CC} = {} & [\gamma_1\gamma_2 k_1 k_2 + \tfrac{1}{2}\gamma_1 k_1 \sinh 2\theta_2 + \tfrac{1}{2}\gamma_2 k_2 \sinh 2\theta_1](k_1^2 - k_2^2)^2 \\
& + \tfrac{1}{4}(k_1^4 + 6k_1^2k_2^2 + k_2^4)\sinh 2\theta_1 \sinh 2\theta_2 \\
& - 4k_1k_2(k_2^2\cosh^2\theta_1 \sinh^2\theta_2 + k_1^2 \sinh^2\theta_1 \cosh^2\theta_2)\,. \qquad (32)
\end{aligned}
$$

The Wronskians for the SS, CS and SC negaton scattering solutions can similarly be obtained.

At this stage, we can state the general result for phase shifts which follows from the centre-of-mass condition. Consider the scattering of any negaton of order n_1 [wave

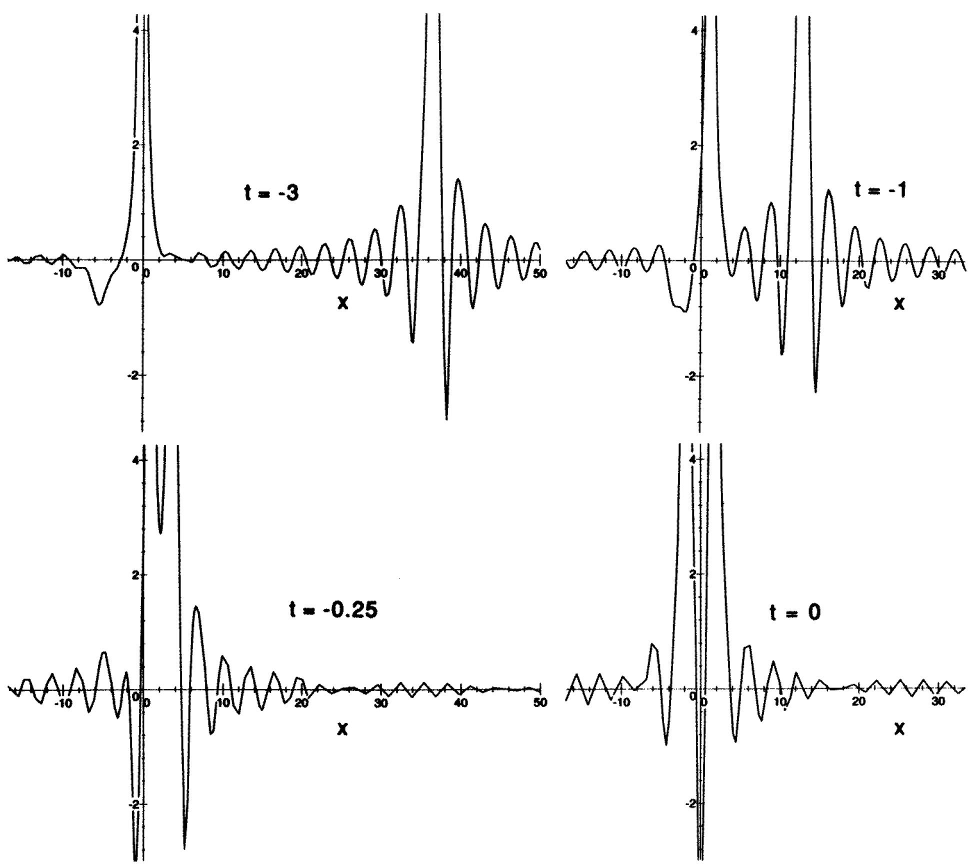

Figure 11. Positon–negaton scattering. The positon $[\tilde{C}]$ has a wave number $\tilde{k} = 1.0$ and the negaton $[C]$ has a wave number $k = 0.5$.

number k_1, mass $M_1 = -4k_1(n_1+1)$] with a negaton of order n_2 [wave number k_2, mass $M_2 = -4k_2(n_2+1)$]. Negaton 1 will undergo a phase shift $\Delta_1 = (n_2+1)\delta$ whereas negaton 2 will have a phase shift $\Delta_2 = -(n_1+1)\delta$, with δ given by (30).

Positon–negaton scattering. Here, we consider the scattering of a $[\tilde{C}]$ positon with negatons of different types. The simplest situation is positon–soliton $[\tilde{C}][C^0]$ scattering. The Wronskian is $W(\tilde{\phi}, \partial_{\tilde{k}}\tilde{\phi}, \phi)$. Matveev [5] has shown that for this case, the soliton has zero phase shift. In our approach, the unchanged phase of the soliton can be immediately and simply understood from the centre-of-mass condition (31) and the fact that the positon $[\tilde{C}]$ has zero mass. The positon phase found by Matveev [5] is

$$\Delta_p = \tfrac{1}{2}\tan^{-1}[2k\tilde{k}/(k^2-\tilde{k}^2)]\,. \tag{33}$$

Proceeding in the same way, the $[\tilde{C}][C]$ Wronskian $W(\tilde{\phi}, \partial_{\tilde{k}}\tilde{\phi}, \phi, \partial_k\phi)$ is given by

$$\begin{aligned} W = {}& (k^2+\tilde{k}^2)^2\big[k\tilde{k}\gamma\tilde{\gamma} + \tfrac{1}{2}k\gamma\sin 2\tilde{\theta} - \tfrac{1}{2}\tilde{k}\tilde{\gamma}\sinh 2\theta\big] + k\tilde{k}(k^2+\tilde{k}^2)[\cosh 2\theta + \cos 2\tilde{\theta}] \\ & -\tfrac{1}{4}(k^4 - 6k^2\tilde{k}^2 + \tilde{k}^4)\sinh 2\theta \sin 2\tilde{\theta} + k\tilde{k}(k^2-\tilde{k}^2)[1+\cosh 2\theta\cos 2\tilde{\theta}]\,. \end{aligned} \tag{34}$$

The scattering process is shown in figure 11. Recall that the positon $[\tilde{C}]$ has zero mass, whereas the negaton $[C]$ has mass $-8k$, twice the mass $-4k$ of a soliton $[C^0]$. Therefore, our centre-of-mass considerations predict that the negaton will have zero phase shift and the positon will have a phase shift $2\Delta_p$, where Δ_p is given by equation (33). We have confirmed this result by careful examination of figure 11. Indeed, we can now state the general result for a positon $[\tilde{C}]$ scattering with any negaton of order n (mass $-4k(n+1)$). This scattering process gives zero phase shift for the negaton and $(n+1)\Delta_p$ for the positon.

6. Singular solutions of mKdV equation

Recently, it has been shown [8] that the concept of negatons and positons can also be extended to the modified KdV equation:

$$v_t - 6v^2 v_x + v_{xxx} = 0\,. \tag{35}$$

If the KdV equation solutions $u(x,t)$ are given by (3) and (10), then the corresponding solution $v(x,t)$ of the mKdV equation is given by [13]

$$v(x,t) = \pm\frac{\partial}{\partial x}\ln\left[\frac{W^*}{W}\right] \tag{36}$$

where

$$W \equiv W(\phi, \partial_k\phi, \ldots) \qquad W^* \equiv W(\phi, \partial_k\phi, \ldots, 1)\,. \tag{37}$$

Thus, given a Wronskian W (and hence u) of the KdV equation, one can immediately obtain the corresponding solution $v(x,t)$ of the mKdV equation by further computing the Wronskian W^*. We would like to point out that it is in fact unnecessary to calculate W^* since it can be shown to be related to W. For example, for the negaton solutions of order n as given by $[C^n]$ and $[S^n]$ one can show that

$$W^*[C^n] = k^{n+1}W[S^n] \qquad W^*[S^n] = k^{n+1}W[C^n]\,. \tag{38}$$

Hence the negaton solutions of order n of the mKdV equation are simply given by

$$v = \pm\frac{\partial}{\partial x}\ln\left[\frac{W[S^n]}{W[C^n]}\right]. \tag{39}$$

The singularities of v come from the zeros of $W[S^n]$ and $W[C^n]$, which have been discussed previously, see (23). Therefore, the mKdV negaton of order n has $(n+1)$ singularities. In particular, there are no non-singular negaton solutions of the mKdV equation!

Using the $[C]$ and $[S]$ negaton Wronskians as given by (7) and (9), we find that the negaton solutions of order 1 of the mKdV equation are given by

$$v(x,t) = \pm \frac{4k(\sinh 2\theta - 2k\gamma \cosh 2\theta)}{(\sinh^2 2\theta - 4k^2\gamma^2)} \,. \tag{40}$$

Note that unlike the $[C^n]$ and $[S^n]$ negatons, the corresponding negatons of the mKdV equation differ from each other simply by a sign. The case of the $n=1$ negaton is plotted in figure 12. We would like to remark here that contrary to the claim of Stahlhofen [8], the negaton (or the corresponding positon) solutions (40) of the mKdV equation do not lead to any new types of solutions of the KdV equation via the Miura transformation

$$u_{1,2} = v^2 \pm v' \tag{41}$$

but as expected, they simply give back the negaton solutions given by equations (17) and (18).

From the negaton–negaton scattering solutions of the KdV equation for wave numbers k_1 and k_2 one finds that

$$W^*_{CC} = k_1^2k_2^2 W_{SS},\, W^*_{SS} = k_1^2k_2^2 W_{CC},\, W^*_{CS} = k_1^2k_2^2 W_{SC},\, W^*_{SC} = k_1^2k_2^2 W_{CS} \tag{42}$$

so that the negaton–negaton scattering solutions of the mKdV equation are given by

$$v = \pm \frac{\partial}{\partial x} \ln\left[\frac{W_{SS}}{W_{CC}}\right], \pm \frac{\partial}{\partial x} \ln\left[\frac{W_{SC}}{W_{CS}}\right]. \tag{43}$$

Similarly, for the case of positon–negaton scattering, we have the relations

$$W^*_{\tilde{C}C} = \tilde{k}^2k^2 W_{\tilde{C}S}(\tilde{\theta} \to \tilde{\theta} + \pi/2)$$

$$W^*_{\tilde{C}S} = \tilde{k}^2k^2 W_{\tilde{C}C}(\tilde{\theta} \to \tilde{\theta} + \pi/2) \tag{44}$$

and hence the positon–negaton scattering solutions of the mKdV equation are given by

$$v = \pm \frac{\partial}{\partial x} \ln\left[\frac{W_{\tilde{C}S}(\tilde{\theta} \to \tilde{\theta} + \pi/2)}{W_{\tilde{C}C}}\right] \tag{45}$$

$$v = \pm \frac{\partial}{\partial x} \ln\left[\frac{W_{\tilde{C}C}(\tilde{\theta} \to \tilde{\theta} + \pi/2)}{W_{\tilde{C}S}}\right]. \tag{46}$$

Finally, for the positon–positon scattering case corresponding to wave numbers $\tilde{k}_1$ and $\tilde{k}_2$, we have the relation

$$W^*_{\tilde{C}\tilde{C}} = \tilde{k}_1^2\tilde{k}_2^2 W_{\tilde{C}\tilde{C}}(\tilde{\theta}_{1,2} \to \tilde{\theta}_{1,2} + \pi/2) \tag{47}$$

so that the positon–positon scattering solution of the mKdV equation is given by

$$v = \pm \frac{\partial}{\partial x} \ln\left[\frac{W_{\tilde{C}\tilde{C}}(\tilde{\theta}_{1,2} \to \tilde{\theta}_{1,2} + \pi/2)}{W_{\tilde{C}\tilde{C}}}\right]. \tag{48}$$

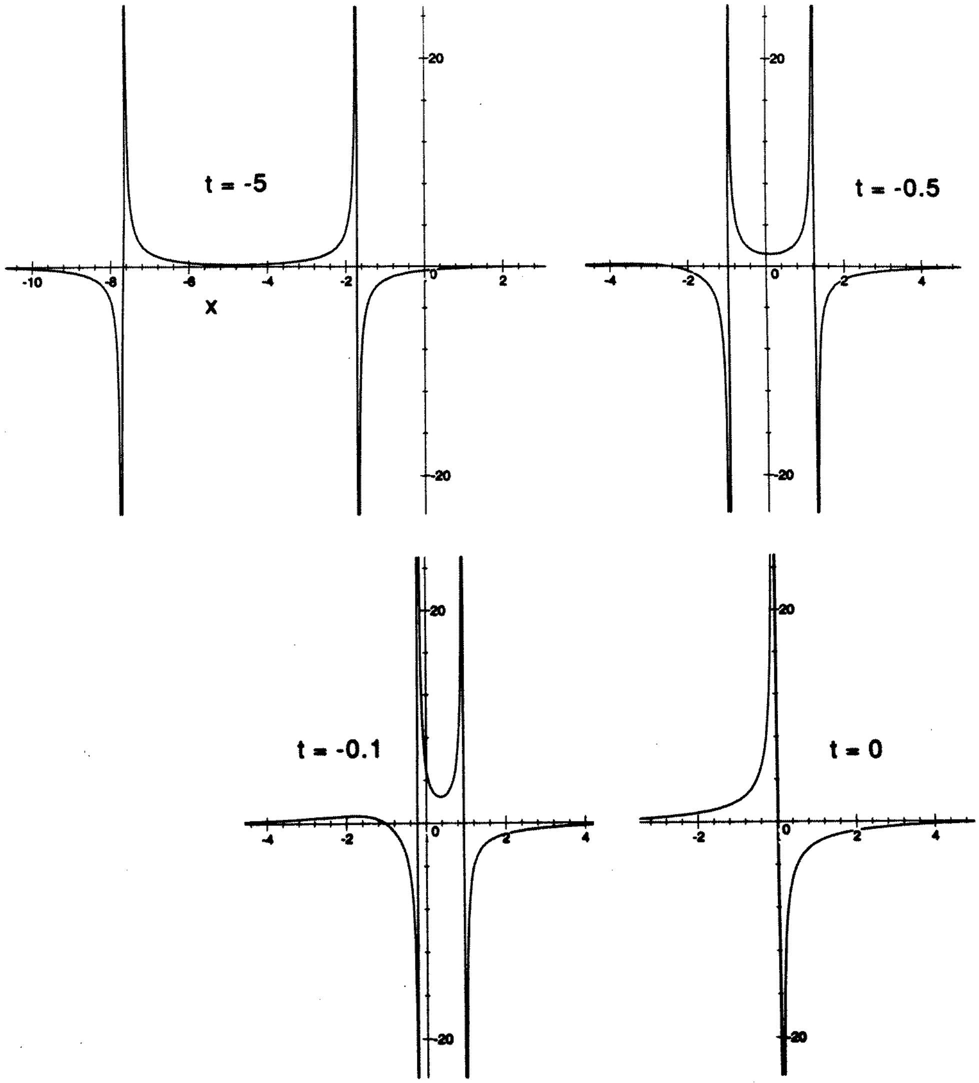

Figure 12. The shape and motion of the order 1 negaton of the mKdV equation as given by (40) for $k = 0.5$ and $\xi(k) = 0$.

7. Singular solutions of KdV and mKdV hierarchy

We wish to point out that the entire discussion so far about negaton and positon solutions can be easily generalized to the entire KdV as well as mKdV hierarchy. To this end, let us first notice that the one-soliton solution of the entire KdV hierarchy is still given by (15) except that for the mth member of the hierarchy, θ is not as given by (4) but by [2]

$$\theta^{(m)} = k(x + \xi(k) - (2k)^{2m} t)\,. \tag{49}$$

As a result, the C and S negaton solutions for the entire KdV hierarchy are again given by (17) and (18) but with θ and γ replaced by $\theta^{(m)}$ and $\gamma^{(m)}$, respectively, where

$$\gamma^{(m)} \equiv \partial_k \theta^{(m)} = x + \xi(x) + k\partial_k \xi(k) - (2m+1)(2k)^{2m} t\,. \tag{50}$$

Thus the shape and motion of the C and S negatons is similar for the entire hierarchy. Besides, all of them have the same mass $-8k$. Using the expressions for $\theta^{(m)}$ and $\gamma^{(m)}$ it is straightforward to work out the Wronskians of order 3 and 4 and obtain W for the entire hierarchy in the case of C^2, S^2, C^3 and S^3 negatons.

Similarly, the $\tilde{C}$ positon solution for the entire hierarchy is again given by (24) but with $\tilde{\theta}$ and $\tilde{\gamma}$ replaced by $\tilde{\theta}^{(m)}$ and $\tilde{\gamma}^{(m)}$, respectively, where

$$\tilde{\theta}^{(m)} = \tilde{k}(x + \tilde{\xi}(k) + (2k)^{2m}t) \tag{51}$$

$$\tilde{\gamma}^{(m)} = x + \xi(k) + k\partial_k\xi(k) + (2m+1)(2k)^{2m}t\,. \tag{52}$$

Similarly, expressions for the higher-order positon solutions can be easily written down and one again finds that the mass of any positon is zero for odd n.

Finally, the results for the scattering of negatons and positons as derived in section 5 can be immediately generalized in the case of the hierarchy by noting that for the KdV hierarchy, as $t \to \pm\infty$, the asymptotic solution for N-solitons is given by

$$u(x,t) = \sum_{i=1}^{N} -2k_i^2 \mathrm{sech}^2(k_i x - k_i(2k_i)^{2m}t \pm \Delta_i) \tag{53}$$

where phase shift Δ is given by (29) i.e. the phase shifts are the same for the entire KdV hierarchy [2]. Thus, the entire discussion in section 5 about the negaton–negaton, positon–negaton and positon–positon scattering goes through for the entire KdV hierarchy and the phase shifts experienced in various collisions are same for any member of the hierarchy.

Similarly, it is clear that the negaton (and positon) solutions for the entire mKdV hierarchy are identical in form to those for the mKdV equation (35) but where θ, γ $(\tilde{\theta}, \tilde{\gamma})$ are to be replaced by $\theta^{(m)}, \gamma^{(m)}$ $(\tilde{\theta}^{(m)}, \tilde{\gamma}^{(m)})$ respectively. The same comments are also applicable to scattering solutions.

8. Conclusions and open problems

In this paper we have discussed in some detail the properties of negaton and positon solutions of the entire KdV hierarchy. Negaton solutions are quite different and at least as interesting as the previously studied positon solutions [5–7]. In particular, there are two distinct types of negaton solutions, whereas there is just one type of positon solution. We have also shown that using the KdV results one can easily obtain the corresponding solutions of the mKdV hierarchy. There are several open problems which are worth investigating. For example, in this paper we have chosen a zero background potential. It would be worthwhile to see if new phenomena arise with non-zero backgrounds. For the case of a constant background potential, there are well known non-singular soliton solutions of the KdV and mKdV equations which tend to non-zero values as $x \to \pm\infty$, and singular solutions of the type described in this paper can be readily constructed. It is expected that these solutions will have some different properties since it will be possible to have negatons moving to the left, in contrast to the situation discussed in this paper. This should provide interesting modifications of the scattering solutions of positons and negatons. Another open problem is to extend this analysis to the Dirac equation [14]. We hope to address these questions in the near future.

Acknowledgments

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